JOURNAL OF MULTIVARIATE ANALYSIS 9, 332–336 (1979)

Counterexample—An Inadmissible Estimator which is Generalized Bayes for a Prior with "Light" Tails

LAWRENCE D. BROWN*

Rutgers University and Cornell University

Communicated by A. Cohen

Previous work on the problem of estimating a univariate normal mean under squared error loss suggests that an estimator should be admissible if and only if it is generalized Bayes for a prior measure, F, whose tail is "light" in the sense that $\int_1^\infty f^{*-1}(x)\,dx = \infty = \int_{-\infty}^{-1} f^{*-1}(x)\,dx$, where f^* denotes the convolution of F with the normal density. (There is also a precise multivariate analog for this condition.) We provide a counterexample which shows that this suggestion is false unless some further regularity conditions are imposed on F.

Consider the much discussed problem of estimating a p-variate normal mean having identity variance-covariance matrix when using squared-error loss. Let F denote a generalized prior (= non-negative measure) over the parameter space, $\Theta = R^p$, of unknown means. Let δ_F denote the corresponding generalized Bayes procedure. Thus

$$\delta_F = \frac{\int \theta\varphi(x-\theta)\,F(d\theta)}{\int \varphi(x-\theta)\,F(d\theta)} = x + \frac{\nabla f^*(x)}{f^*(x)}, \tag{1}$$

where φ denotes the standard normal density and

$$f^*(x) = \int \varphi(x-\theta)\,F(d\theta).$$

It is known (Brown [1, Theorem 3.1.1]) that every admissible procedure is of this form with

$$f^*(x) < \infty \qquad \forall x \in R^p. \tag{2}$$

Received July 11, 1978; revised November 14, 1978.

AMS 1970 subject classification: Primary 62E15.

Key words and phrases: estimating a normal mean, generalized Bayes estimators.

* Work supported in part by the National Science Foundation under Grant MPS 72-05075–A02.

0047-259X/79/020332–05$02.00/0

Not all procedures of the form δ_F are admissible. When $p = 1$ we say that F has a *heavy* tail if

$$\int_1^{\infty} (1/f^*(x))\, dx < \infty \qquad \text{or} \qquad \int_{-\infty}^{-1} (1/f^*(x))\, dx < \infty. \tag{3}$$

If the tail of F is not heavy then it is *light*. More generally ($p \geqslant 1$), the tail of F is "heavy" if the differential equation $\nabla \cdot (f^*\nabla k) = 0$ has a bounded non-constant, differentiable solution on $\{x\colon \| x \| \geqslant 1\}$ satisfying $k(x) = 1$, $\| x \| = 1$. In Brown [1, p. 885] it is shown that δ_F is inadmissible if the tails of F are heavy.

Stein's method. A more elementary proof of this fact proceeds, in outline, as follows: Let δ_F be a generalized Bayes procedure with everywhere finite risk. Stein [5] has shown that if $\delta(x) = \delta_F(x) + \nabla k(x)/k(x)$ for some continuously differentiable function, $k > 0$,

$$E_\theta(\| \delta_F - \theta \|^2) - E_\theta(\| \delta - \theta \|^2) = E_\theta[(-2\nabla \cdot (f^*\nabla k))/f^*k + \| \nabla k/k \|^2] \tag{4}$$

so long as $E_\theta(\| \nabla k/k \|^2) < \infty$, $\forall \theta \in R^p$. The condition that the tail of F be heavy is *equivalent* to the existence of a suitable function k which is subharmonic (relative to f^*) in the sense that

$$\nabla \cdot (f^*\nabla k) \leqslant 0 \qquad \forall x \in R^p. \tag{5}$$

Obviously, when such a function, k, exists the expression (4) shows that δ_F is inadmissible.

The major part of Brown [1] is devoted to establishing the converse result: δ_F is admissible if the tail of F is light. This result is established there only under certain mild additional regularity conditions. (These conditions are slightly weaker than the condition that δ_F have uniformly bounded risk.) At the time these conditions appeared to us to be merely technical ones, required by certain inadequacies of our method of proof. Indeed, Srinivasan [4] has recently proved this admissibility result under notably milder conditions than ours. More compelling yet, mathematical aesthetics appeal for a reversal of the elegant argument involving Stein's method.

Unfortunately this appeal cannot be satisfied. The following counterexample shows that there do exist priors with light tails for which the generalized Bayes estimator is inadmissible. Naturally the prior, and corresponding procedure, are weird; for the results of Brown and Srinivasan show convincingly that any such counterexample must be peculiar.

Contruction. Let $p = 1$. The prior, F, will put mass π_i on the point a_i, $i = 0, \pm 1, \pm 2,\ldots$, where $\pi_i = \pi_{-i}$, and $a_{-i} = -a_i$. The values a_i, π_i, $i = 0, 1,\ldots$ are constructed by induction along with values ρ_{i-1} as follows:

Let $a_0 = 0$, $\pi_0 = 3$, and let $\rho_{-1} > 0$ satisfy $\pi_0 \varphi(\rho_{-1}) = 1/10$. Note that

$\rho_{-1} > 2$. Given a_i, π_i, and ρ_{i-1} inductively define $\rho_i = (1 - \Phi(2\rho_{i-1} + 3))^{-1/2} + 2$ where Φ denotes the standard normal C.D.F., and

$$a_{i+1} = a_i + \rho_{i-1} + 1 + \rho_i, \qquad \text{and}$$

$$\pi_{i+1}^{-1} = 10\varphi(\rho_i).$$

For convenience let $b_i = a_i + \rho_{i-1}$. Note that $\pi_i\varphi(b_i - a_i) = 1/10$ and $a_{i+1} = b_i + 1 + \rho_i$. See Figure 1.

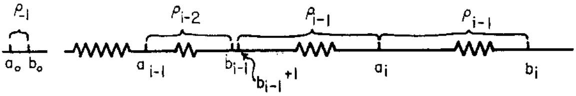

FIG. 1. Relative positions of a_i, b_i, ρ_i.

Verification. It must be shown that $f^*(x) < \infty$, $\forall x \in R$, that the tails of F are light, and that δ_F is inadmissible. This verification will be carried out in several steps.

Briefly, it will be shown that $f^*(x)$ is small for $b_i < |x| < b_i + 1$ $(i \geqslant 0)$ which implies that $f^*(x) < \infty$ and that the tails of F are light. Then it will be shown that $\delta_F(x)$ is very near a_i when $b_{i-1} + 2 < x < b_i - 1$. Thus if, for example, $\theta = a_i$ $(i \geqslant 1)$ then δ_F is very near θ with probability nearly one; but there is a very small probability that $x < b_{i-1} - 1$ or $x > b_i + 2$ and when this occurs $|\delta_F(x) - a_i|$ is approximately $\rho_i + \rho_{i-1} + 1$, and ρ_i is so huge that overall $\mathbf{R}(\theta, \delta_F) > 1$. Similar reasoning extends to all values of θ and will be made precise to show that $\mathbf{R}(\theta, \delta_F) > 1$, $\forall \theta \in R$, and hence δ_F is inadmissible.

I. $\pi_i\varphi(x - a_i) < (1/10)\exp\{-|x - a_i|(|x - a_i| - \rho_{|i|-1})/2\} < (1/10)\exp\{-(|x - a_i| - \rho_{|i|-1})^2/2\}$ if $|x - a_i| > \rho_{|i|-1}$; $i = 0, \pm 1, \ldots$.

Proof. Suppose $x > 0$, then $\pi_i\varphi(x - a_i) = (1/(2\pi)^{1/2})\pi_i\exp\{-(x - c + c - a_i)^2/2\} = (1/(2\pi)^{1/2})\pi_i\exp\{-(c - a_i)^2/2\}\exp\{-(x - c)(c - a_i) - (x - c)^2/2\} \leqslant (1/(2\pi)^{1/2})\pi_i\exp\{-(c - a_i)^2/2\}\exp\{-(x - a_i)(x - c)/2\}$ if $(x - c)(c - a_i) \geqslant 0$. Choose $c = a_i + \rho_{|i|-1}$ if $x > a_i$, or $c = a_i - \rho_{|i|-1}$ if $x < a_i$, and apply the above while noting that $(1/(2\pi)^{1/2})\pi_i\exp\{-(c - a_i)^2/2\} = \pi_i\varphi(\rho_{|i|-1}) = 1/10$. The result for $x < 0$ follows by symmetry. ▌

II. *If* $b_i < |x| < b_{i+1}$, $i = 0, 1, \ldots$, *then* $f^*(x) < 2/5$.

Proof. The following is for $x > 0$. The proof for $x < 0$ is symmetric. $f^*(x) = \sum_{j=-\infty}^{\infty} \pi_j\varphi(x - a_j) = \sum_{j=1}^{\infty} \pi_{i+j}\varphi(x - a_{i+j}) + \sum_{j=0}^{\infty} \pi_{i-j}\varphi(x - a_{i-j})$. Note that $a_i - a_{i-1} = \rho_{i-2} + 1 + \rho_{i-1} > 1$ (actually, $\gg 1$), for $i \geqslant 1$. Hence $\sum_{j=1}^{\infty} \pi_{i+j}\varphi(x - a_{i+j}) \leqslant (1/10)\sum_{j=1}^{\infty}\exp[-(j - 1)^2/2] < 1/5$ by I. Similarly $\sum_{j=0}^{\infty} \pi_{i-j}\varphi(x - a_{i-j}) < 1/5$. ▌

We will later also need:

III. *If* $b_{i-1} + 2 < x < b_i - 1$ *then* $\sum_{j=i+1}^{\infty} \pi_j \varphi(x - a_j) < 1/10$ *and* $\sum_{j=-\infty}^{i-1} \pi_j \varphi(x - a_j) < 1/10$.

Proof. Similar to II. ▮

PROPOSITION 1. $f^*(x) < \infty\ \forall x \in R$, *and the tails of* F *are light.*

Proof. Note that $\{x: f^*(x) < \infty\} \supset \{x: b_i < |x| < b_i + 1\}$ by II and $\{x: f^*(x) < \infty\}$ is convex; see e.g. Lehmann [3, p. 31]. Hence $f^*(x) < \infty\ \forall x \in R$. By II $\int_{-B}^{-1} (1/f^*(x))\, dx = \int_1^B (1/f^*(x))\, dx > (5/2)\{\max i\colon b_i + 1 < B\} \to \infty$ as $B \to \infty$. Hence the tails of F are light by (3). ▮

IV. *If* $x > b_{i-1} + 1$ $(i \geqslant 1)$ *then* $\sum_{j=-\infty}^{i-1} (x - a_j)\, \pi_j \varphi(x - a_j) < 2/5$. *If* $x < b_i$ $(i \geqslant 0)$ *then* $\sum_{j=i+1}^{\infty} (a_j - x)\, \pi_j \varphi(x - a_j) < 2/5$.

Proof. If $x > b_{i-1} + 1$ and $j \leqslant i - 1$ then $(x - a_j)\, \pi_j \varphi(x - a_j) < (1/10)(x - a_j) \exp[-\rho_{|j|-1}(x - a_j - \rho_{|j|-1})/2 - (x - a_j - \rho_{|j|-1})^2/2]$ by I. But, $x - a_j - \rho_{|j|-1} > 1$ and $\rho_{|j|-1} > 1$ so that $(x - a_j) \exp(-\rho_{|j|-1}(x - a_j - \rho_{|j|-1})/2)$ $\leqslant \max\{t \exp(-c(t - c)/2)\colon c, t \ni 1 \leqslant c \leqslant t - 1\} = 2e^{-1/2} < 2$. Thus $\sum_{j=-\infty}^{i-1} (x - a_j)\, \pi_j \varphi(x - a_j) < (1/10) \sum_{j=-\infty}^{i-1} 2 \exp(-(x - a_j - \rho_{|j|-1})^2/2) < 2/5$ as in the proof of II. The second assertion of IV is proved similarly. ▮

V. *If* $b_{i-1} + 2 < x < b_i - 1$ $(i \geqslant 1)$ *then* $\pi_i \varphi(x - a_i) > (1/10)\, e^{(\rho_{i-1} - 1/2)} > 2/5$.

Proof. $10\pi_i \varphi(x - a_i) = \pi_i \varphi(x - a_i)/\pi_i \varphi(\rho_{i-1}) = \exp\{(\rho_{i-1}(\rho_{i-1} - |x - a_i|) - (\rho_{i-1} - |x - a_i|)^2/2)\} \geqslant \exp(\rho_{i-1} - 1/2)$ since the condition on x is equivalent to $(\rho_{i-1} - |x - a_i|) \geqslant 1$. The second inequality follows since $\rho_{i-1} > 2$. ▮

VI. *If* $b_{i-1} + 2 < x < b_i - 1$ $(i \geqslant 1)$ *then* $|\delta_F(x) - a_i| \leqslant 2$.

Proof. $\delta_F(x) - a_i = \sum_{j=-\infty}^{\infty} (a_j - a_i)\, \pi_j \varphi(x - a_j) / \sum_{j=-\infty}^{\infty} \pi_j \varphi(x - a_j)$. Hence

$$
\begin{aligned}
(\delta_F(x) - a_i)^+ &\leqslant \sum_{j=i+1}^{\infty} (a_j - a_i) \frac{\pi_j \varphi(x - a_j)}{\pi_i \varphi(x - a_i)} \\
&= \sum_{j=i+1}^{\infty} (a_j - x + x - a_i) \frac{\pi_j \varphi(x - a_j)}{\pi_i \varphi(x - a_i)} \\
&\leqslant (2/5)^{-1} \sum_{j=i+1}^{\infty} (a_j - x)\, \pi_j \varphi(x - a_j) + (1/10)\, \rho_{i-1}/\pi_i \varphi(x - a_i)
\end{aligned}
$$

since $\pi_i \varphi(x - a_i) > 2/5 > 1/10 > \sum_{j=i+1}^{\infty} \pi_j \varphi(x - a_j)$ by V and III along with $x - a_i < \rho_{i-1}$. Applying IV and V yields

$$(\delta_F(x) - a_i)^+ < (5/2)(2/5) + \rho_{i-1} e^{-\rho_{i-1} + 1/2} < 2.$$

A similar inequality applies to $(\delta_F(x) - a_i)^-$. ▮

PROPOSITION 2. $\mathbf{R}(\theta, \delta_F) = E_\theta((\delta_F(X) - \theta)^2) > 1 \; \forall \theta \in R$. *Hence δ_F is inadmissible.*

Proof. Let $b_{i-1} \leqslant \theta \leqslant b_i$ $(i \geqslant 1)$. Then $Pr_\theta(X > b_i + 2) = 1 - \Phi(b_i + 2 - \theta) \geqslant 1 - \Phi(2\rho_{i-1} + 3)$. Furthermore, when $x > b_i + 2$ then $(\delta_F(x) - \theta)^2 = ((a_{i+1} - \theta) + (\delta_F(x) - a_{i+1}))^2 \geqslant ((\rho_i + 1) - 2)^2 = (\rho_i - 1)^2 > (\rho_i - 2)^2$ by VI and the monotonicity of δ_F. Thus

$$\mathbf{R}(\theta, \delta_F) > (\rho_i - 2)^2(1 - \Phi(2\rho_{i-1} + 3)) \geqslant 1.$$

The same basic argument applies when $-b_0 \leqslant \theta \leqslant b_0$. Finally, $\mathbf{R}(\theta, \delta_F) = \mathbf{R}(-\theta, \delta_F)$ by symmetry. Hence $\mathbf{R}(\theta, \delta) > 1 \; \forall \theta \in R$. The minimax estimator, defined by $\delta(x) = x$, has $\mathbf{R}(\theta, \delta) \equiv 1$. Hence $\mathbf{R}(\theta, \delta) < \mathbf{R}(\theta, \delta_F) \; \forall \theta \in R$ and δ_F is inadmissible. ▌

REFERENCES

[1] BROWN, L. D. (1971). Admissible estimators, recurrent diffusions, and insoluble boundary value problems. *Ann. Math. Statist.* 42 855–903.

[2] BROWN, L. D. (1973). Correction to "Admissible estimators, recurrent diffusions, and insoluble boundary value problems." *Ann. Statist.* 1 594–596.

[3] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

[4] SRINIVASAN, C. (1978). Admissible generalized Bayes estimators and exterior boundary value problems. Purdue University, Dept. of Statist. Mimeograph series No. 78–2.

[5] STEIN, C. (1973). Estimation of the mean of a multivariate normal distribution. In *Proc. Prague Symp. Asymptotic Statist.*, pp. 345–381.